JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **171**, 377–388 (1992)

Bifurcation Properties for a Sequence of Approximation of Delay Equations

OVIDE ARINO

Université de Pau, France

AND

RACHID BENKHALTI

Pacific Lutheran University, Tacoma, Washington

Submitted by Stavros N. Busenberg

Received July 2, 1990

I. INTRODUCTION

Many works have been devoted to the study of equations of the type

$$\frac{dx}{dt} = -\lambda f(x(t), x(t-1)), \tag{1}$$

where $\lambda > 0$, and $(\partial/\partial y) f(x, y) > 0$ for $x \neq 0$ (see [4, 10]), or the even more restricted equation, $dx/dt = -\lambda f(x(t-1))$.

The outstanding fact established, for the first time by Wright [10], is the existence of a class of slowly oscillating solutions for (1) in which the trivial solution ($x \equiv 0$) appears as an equilibrium whose stability changes with λ. This class determines a convex cone of initial data invariant under the equation. And the change of equilibrium, the bifurcation to new equilibria can be expressed through a family of Poincaré operators on the cone.

In contrast, little has been done regarding equations of the type

$$\frac{dx}{dt} = \lambda f(x(t), x(t-1)), \tag{2}$$

with the same hypotheses of f and the sign of λ. Such equations were studied recently by H. L. Smith in relation to the notion of "cooperativity." The monotonicity with respect to the delay term leads to a monotonicity property of the semigroup solution, or to the positivity of the semigroup associated with the linearized equation around trivial solutions.

0022-247X/92 $5.00

409/171/2-6

Based on recent results by M. H. Hirsch [6], Smith established some very interesting general properties, particularly the existence of curves of monotone solutions connecting trivial solutions. On the other hand, Smith was not interested in the oscillation, nor in the relationship between oscillation and stability. More precisely, this aspect was put aside after he recalled a general result of Hirsh indicating that if a Hopf bifurcation occurs (which is the appearance of oscillating periodic solutions), then this phenomenon does not lead to stabilization. In other words, the system cannot gain stability by Hopf bifurcation.

In spite of this, the oscillatory properties of equations of type (2) can be of interest in modeling real phenomena. Thus the equation

$$\frac{dx}{dt} = -x(t) + x^2(t-\omega), \tag{3}$$

appears in the study of electronic cascade [2]. The interesting steady state here is the solution $x \equiv 1$, and we are looking for perturbations of this state which will lead to solutions oscillating around 1.

It appears that we are in the presence of a very sensitive phenomenon: numerically, the oscillations can only be maintained for a very short time; the solutions, then, reach either zero as an exponential, or infinity at a superexponential rate.

In [2] O. Arino and P. Seguier established a number of results on certain equations of type (2):

— the existence of oscillating solutions,

— the convergence of these solutions to an equilibrium state for certain values of the parameter,

— the existence of undamped oscillating solutions in the case of a very small nonlinearity,

— and the existence of periodic solutions of small amplitude by Hopf bifurcation.

In [3] R. Benkhalti established the existence of a global branch of Hopf bifurcation (see also [1]). But the question of the stability of these solutions in the set of oscillatory solutions and the stability of the trivial solution before the bifurcation point in the solution set are still open.

Here, we present an approach which attempts to answer these questions. To approach consists in substituting a discrete family of approximations for the initial equation, based on the Euler method.

In Section 2, we show that the equations obtained in this way share some of the properties of the initial equation, notably the monotonicity, which leads to the existence of oscillation. In fact, the oscillating solutions are approximations of oscillating solutions for the differential equation.

From our point of view, this fact justifies the study of bifurcation phenomena for the approximations. The results presented in Section 3 are disappointing but are not a surprise. The Hopf bifurcation for the differential equation is not revealed on the level of the chosen approximations. In particular, we establish in Theorem III.2, that, for approximations with odd order greater than 3, no bifurcation phenomenon of any kind occurs near the trivial solution. As a concluding remark this result can be compared with that of Hofbauer and Iooss [7]; the idea of those authors is different from ours in the sense that the differential equation does not present a bifurcation. A parameter is introduced in the approximations. The differential equation is then replaced by a one parameter family of transformations, and they focus on the link between the existence of periodic solutions of the linearized differential equations and the appearance of an invariant curve for the transformations.

However, the two studies are inspired by the same preoccupation shared by an increasing number of researchers today (see, in another domain the results of Hankerson and Peterson [5]), to give to the qualitative study of discretized equations a completely separate status.

Comparing qualitative properties of a differential equation to those of its approximations is in fact a natural thing to do. It has also been pursued for the case of the logistic equation, and has given one of the main examples of "spurious solution." Recently, F. Kappel and K. Schmitt [8] gave a detailed study of approximation of a delay equation by ordinary differential equations which seem to inherit nearly all the properties of the delay equation.

II. Properties of the Differential Equation and its Approximations

II.a. *The Differential Equation*

Consider the differential equation

$$\frac{dx}{dt} = \lambda f(x(t), x(t-1)), \tag{2}$$

where λ is a positive parameter, f is $C^2(\mathscr{R}^2, \mathscr{R})$, and f is Lipschitz continuous in the first variable with constant M.

In [1], under the following hypotheses, $f(0,0)=0$, $(\partial/\partial v) f(u,v) > 0$, $|a| = |(\partial/\partial u) f(0,0)| < b = (\partial/\partial v) f(0,0))$, and some other technical assumptions, we proved:

PROPOSITION II.1. *For each* $\lambda_1 > 0$, *there exists a real number* $B > 0$ *such that if* x *is a periodic solution of* (2), *oscillating near to zero, and if* $0 < \lambda \leqslant \lambda_1$, *then* $|x| \leqslant B$.

This result is the best possible in terms of a priori estimates, and it is also valid for oscillating solutions, which are not periodic. Then it has to be stated as an asymptotic result

$$\limsup_{t \to \infty} |x(t)| \leqslant B.$$

Let $C = C([0, 1], \mathscr{R})$. Let $\mathscr{S}$ denote the closure in $\mathscr{R}^+ \times \mathscr{R}^+ \times C$ of the set of triples (λ, T, φ) such that φ is a datum of a T-periodic solution of (2), and let $\mathscr{S}_1$ be the maximal connected component of $(\lambda_1, T_{1,1}, 0)$ in $\mathscr{S}$, where $T_{1,1}$ is the initial period [1].

THEOREM II.2. *The set* $\mathscr{S}_1$ *contains a connected set* $\mathscr{C}_0$ *such that* $\mathscr{C}_0 \cap \mathscr{R}^+ \times \mathscr{R}^+ \times \{0\} = (\lambda_1, T_{1,1}, 0)$. *Also,* $\mathscr{C}_0$ *is unbounded in* λ.

II.b. *Approximation*

Let us write (2) as $dx/dt = f(x(t), x(t-\omega))$, where $\omega = \lambda$ and let $\omega = ph$ where p is an integer and h is the step size.

Applying the Euler scheme, we obtain

$$\frac{x_n - x_{n-1}}{h} = f(x_n, x_{n-p}),$$

$$x_n = \frac{1}{1-ah}(x_{n-1} + hbx_{n-p}) + \frac{h}{1-ah} g(x_n, x_{n-p}), \tag{4}$$

where g is a nondecreasing map such that

$$g(x, y) = f(x, y) - ax - by.$$

DEFINITION II.3. A solution (x_n) of (4) is said to be oscillating if, for every n, there exist $n_1 > n$, and $n_2 > n$ such that $x_{n_1} \geqslant 0$ and $x_{n_2} \leqslant 0$.

PROPOSITION II.4. *Let* (x_n) *and* (y_n) *satisfy* (4). *If the initial conditions are such that* $x_{-j} \leqslant y_{-j}$ *for all* $j \in [1, p]$, *then* $x_n \leqslant y_n$ *for all* $n \geqslant 0$.

Proof. Assume that $x_0 > y_0$. Then we have

$$\begin{aligned} y_0 - x_0 &= y_{-1} - x_{-1} + h(f(y_0, y_{-p}) - f(x_0, x_{-p})) \\ &= y_{-1} - x_{-1} + h(f(y_0, y_{-p}) - f(y_0, x_{-p}) \\ &\quad + f(y_0, x_{-p}) - f(x_0, x_{-p})), \end{aligned}$$

therefore,

$$|y_0 - x_0| < h\,|f(y_0, x_{-p}) - f(x_0, x_{-p})|$$
$$< hM\,|y_0 - x_0| < |y_0 - x_0|,$$

which is a contradiction.

PROPOSITION II.5. *A necessary and sufficient condition for a solution* (x_n) *of* (4) *to oscillate is that for every* n, *we have*

$$\max_{1 \leqslant j \leqslant p} x_{n-j} \geqslant 0 \qquad \text{and} \qquad \min_{1 \leqslant j \leqslant p} x_{n-j} \leqslant 0.$$

The above proposition expresses the fact that if at some n_0 we can find p successive terms that are either greater or less than 0, then this solution is not oscillating.

Now, let us construct oscillating solutions of (4).

PROPOSITION II.6. *Let* (y_{-j}), (z_{-j}) *be two data such that* $y_{-j} < 0 < z_{-j}$, *such that the vectors* (y_{-j}) *and* (z_{-j}) *are independent. Let* $x_{-j}(\Theta) = \Theta y_{-j} + (1-\Theta) z_{-j}$, *for* $j \in [1, p]$, *and* $\Theta \in [0, 1]$. *Then, there exists* $\Theta \in [0, 1]$ *such that the sequence* $x_n(\Theta)$, *with initial data* $x_{-j}(\Theta)$ *as above, is an oscillating solution of* (4).

Proof. Define the sets

$$A^+ = \{\theta \in [0, 1]\text{: there exists } n_0 \text{ such that } \min_{1 \leqslant j \leqslant p} x_{n_0 - j}(\theta) > 0\}$$

$$A^- = \{\theta \in [0, 1]\text{: there exists } n_0 \text{ such that } \max_{1 \leqslant j \leqslant p} x_{n_0 - j}(\theta) < 0\}.$$

First, observe that A^+ and A^- are nonempty sets, for $0 \in A^+$ and $1 \in A^-$. Also $A^+ \cap A^-$ is empty. Second, note that A^+ and A^- are open sets, consequently they cannot cover $[0, 1]$, since $[0, 1]$ is a connected set. Therefore all $x_{-j}(\theta)$, for $\theta \in [0, 1]/(A^+ \cup A^-)$ provide initial data of oscillating solutions of (4). We now show A^+ and A^- are open.

Consider, for every whole number n, the maps

$$\tau_n\colon [0, 1] \to \mathscr{R}$$
$$\theta \to \min_{1 \leqslant j \leqslant p} x_{n-j}(\theta)$$
$$\eta_n\colon [0, 1] \to \mathscr{R}$$
$$\theta \to \max_{1 \leqslant j \leqslant p} x_{n-j}(\theta).$$

Note that τ_n and η_n are continuous for all n. Furthermore we have $A^+ = \bigcup_n (\tau_n^{-1}(]0, \infty[))$, $A^- = \bigcup_n (\eta_n^{-1}(]-\infty, 0[))$, which are open.

The following two lemmas give a priori estimates and are needed to prove the convergence of the scheme.

We assume that there exists $M' \geqslant M$ so that $f(x, y) + M'x \geqslant \mu > -\infty$, for every $x \geqslant 0$, y in $\mathscr{R}$, and μ a real number.

Remark. This assumption is not implied by the Lipschitz character of f. In fact, assuming f be Lipschitz continuous is equivalent to assuming the extra condition

$$\inf_{y \in \mathscr{R}} f(0, y) > -\infty.$$

Set $g(x, y) = f(x, y) + M'x$. Equation (4) reads as

$$x_n(1 + hM') = x_{n-1} + hg(x_n, x_{n-p}).$$

We will prove first that the oscillating solutions are bounded above by some constant independent of the solution.

LEMMA II.7. *For every solution $(x_n)_{n \in \mathcal{N}}$ which oscillates, we have*

$$x_n \leqslant (p+1)(h\,|\mu| + 1)(1 + hM')^{p+1} = K_{p'} \qquad \textit{for} \quad n \geqslant 0.$$

Proof. What we shall prove is that if for some $n_0, x_{n_0} > K_p$, then $x_{n_0+1} > K_{p-1}$. Note that x_{n_0+1} is the root of the equation $\varphi(u) = x_{n_0}$, where $\varphi(u) = u(1 + hM') - hg(u, x_{n_0-p+1})$. For h small enough, the function φ is increasing. If we can prove that $\varphi(0, \infty) \ni x_{n_0}$, then it will imply that x_{n_0+1} is a positive number. Because of the assumption, we have for $u \geqslant 0$, $\varphi(u) \leqslant u(1 + hM') - h\mu$, which yields $\varphi(0) \leqslant -h\mu$.

On the other hand, the value at infinity is determined by $u(1 + hM')$. We have $\varphi(\infty) = \infty$. Therefore, the desired result will follow if we take $x_{n_0} > |h\mu|$.

If $u > 0$, we have an estimate of u in terms of x_{n_0}

$$u \geqslant \frac{x_{n_0} - |h\pi|}{1 + hM'}.$$

Therefore, if $x_{n_0} \geqslant K_p$, we have

$$x_{n_0} \geqslant p(h\,|\mu| + 1)(1 + hM')^{p+1} + h\,|\mu|,$$

and so

$$x_{n_0} - h\,|\mu| \geqslant p(h\,|\mu| + 1)(1 + hM')^{p+1}.$$

Therefore, we obtain $x_{n_0+1} \geqslant p(h\,|\mu| + 1)(1 + hM')^p = K_{p-1}$. We can proceed by induction to obtain estimates for $x_{n_0}, ..., x_{n_0+(p-1)}$. Having then

p successive elements of the same sign, we know that the solution does not oscillate, which completes the proof of the lemma.

LEMMA II.8. *For every solution* (x_n) *which oscillates,* x_n *is bounded below. Moreover, for* $n \geqslant 2p$, *we have* $x_n \geqslant m$, *for some* m *independent of the solution.*

Proof. Let (x_n) be an oscillating solution. We know that $x_n \leqslant A$, for $n \geqslant 0$, where $A \leqslant K_p$ (defined in the previous lemma). So, we have the following estimate $x_n \leqslant x_{n-1} + hf(x_n, A)$, $n \geqslant p$. We will obtain the desired estimate by looking backward at this inequality. Starting from an index n, $n \geqslant p$, for which $x_n \geqslant 0$, we have

$$x_{n-1} \geqslant x_n - h(f(x_n, A) - f(0, A)) - hf(0, A),$$

but, for h small enough, f being Lipschitz continuous in x we have $x - hf(x, A) \geqslant 0$ for $x \geqslant 0$ and so

$$x_{n-1} \geqslant -hf(0, A).$$

Denote $m_1 = -hf(0, A)$. If $n \geqslant p+1$, we can use this estimate of x_{n-1} to derive an estimate for x_{n-2}

$$x_{n-2} \geqslant x_{n-1} - h(f(x_{n-1}, A) - f(m_1, A)) - hf(m_1, A).$$

This gives

$$x_{n-2} - m_1 \geqslant -hf(m_1, A).$$

Define inductively

$$m_j = m_{j-1} - hf(m_{j-1}, A).$$

As long as $n \geqslant p + j - 1$ and $x_n \geqslant 0$, we obtain $x_{n-j} \geqslant m_j$. Since we know that x_n takes a nonnegative value at least once on each family of p consecutive indices, we can conclude the proof of the lemma. We have $x_n \geqslant m = \min_{j=1,\dots,p} m_j$, for $n \geqslant 2p$.

THEOREM II.9. *In any segment* $[\varphi, \psi] = \{\lambda\varphi + (1-\lambda)\psi : 0 \leqslant \lambda \leqslant 1\}$, $\varphi \leqslant 0 \leqslant \psi$, *there is at least an initial value of an oscillating solution whch is the limit of a sequence of initial values for the approximating equations.*

Proof. Let X_0 be an oscillating solution of (2).

Consider an approximation of X_0 obtained by taking the values of X_0 at n points. Assume that n is large enough so that the solution of (4) starting from the approximation is oscillating on an interval $[0, T]$, T given arbitrarily.

Now there exist $\varphi_1 \leqslant 0 \leqslant \varphi_2$ such that $X_0 = \theta\varphi_1 + (1-\theta)\varphi_2$. Denote by $\varphi_i^{(p)}$ the approximation of φ_i obtained by taking $\varphi_{i,j}^{(p)} = \varphi_i(j\omega/p)$, $1 \leqslant j \leqslant p$, $i = 1, 2$. So there exists a convex combination of $\varphi_1^{(p)}$ and $\varphi_2^{(p)}$ which leads to an oscillating solution, $\theta_p \varphi_1^{(p)} + (1-\theta_p)\varphi_2^{(p)}$.

Claim. $\theta_p \to \theta$ as $p \to \infty$.

In fact suppose this is not true, so that there exists a sequence $p_k \to \infty$, with $|\theta_{p_k} - \theta| \geqslant \eta > 0$.

Let us compare $X_0^{(p_k)}$ with $\theta_{p_k}\varphi_1^{(p_k)} + (1-\theta_{p_k})\varphi_2^{(p_k)}$.

The difference is $(\theta - \theta_{p_k})(\varphi_1^{(p_k)} - \varphi_2^{(p_k)})$ and the minimum of its absolute value is greater or equal to $\eta(\min |\varphi_1| + |\varphi_2|)$.

On the other hand, as $k \to \infty$, it follows from the properties of approximations that the solution associated with $X_0^{(p_k)}$ will stay close to the exact solution on an interval going to ∞. This means that, for any $T > 0$, there exists N such that $k \geqslant N$ implies the solution starting from $X_0^{(p_k)}$ keeps oscillating on $[0, T]$. So, in view of the proof of Lemmas II.7 and II.8, we have

$$m \leqslant X_{0,j}^{(p_k)} \leqslant M, \qquad \text{for } \frac{j_\omega}{p_k} < T.$$

But, this together with the fact that the other solution is either less or equal or greater or equal to $X_{0,j}^{(p_k)}$ and also oscillating implies that

$$|X_{0,j}^{(p_k)} - (\theta_{p_k}\varphi_1^{(p_k)} + (1-\theta_{p_k})\varphi_2^{(p_k)})_{(j)}| \geqslant \left(\frac{1+hA}{1+hB}\right)^{j\omega/pk_\eta}.$$

This leads to a contradiction as $T \to \infty$.

Remark. Let X denote an oscillating solution of (2), and $(X_n^{(h)})$ be the approximations obtained using (4). Then the approximations $X_n^{(h)}$ will converge to X, as $n \to \infty$ and $h \to 0$, if we have uniqueness of oscillating solutions. In fact, we shall have convergence if we cannot find two oscillating solutions that can be compared (one is less than the other). This is true, for example, if Eq. (2) has no eigenvalues with positive real part other than the real one, see [2].

III. Result

Let F be the operator given by

$$F: \mathscr{R}^+ \times \mathscr{R}^p \to \mathscr{R}^p$$

$$(h, x_{-1}, ..., x_{-p}) \to (x_0, x_{-1}, ..., x_{-p+1}),$$

where

$$x_0 = \frac{1}{1-ah}(x_{-1} + bhx_{-p}) + hg(x_0, x_{-p}). \tag{5}$$

We are looking for fixed points of F, that is

$$X = F(h, X). \tag{6}$$

Note that $X \equiv 0$ is a trivial solution of (6) for evry h. We are interested in solutions arising near to zero as a result of a bifurcation process.

A. *Rabinowitz Bifurcation*

The characteristic polynomial associated with the linearization of (6) around 0 is $p_h(r) = (1-ah)r^p - r^{p-1} - bh$. Note $p_h(1) = -(a+b)h < 0$, since $b > |a|$. So $r = 1$ is a root of p_h, it is then simple when $h = 0$. Since $(\partial/\partial h)\, p_h(1) = -(a+b) \neq 0$ for all h, we conclude that $(0, 0)$ is a bifurcation point and the bifurcating branch is unbounded (see [1]). But, if we let $h = 0$ in (5), we obtain $x_0 = x_{-1}$, and it follows that $X = (x_{-1}, ..., x_{-1})$ is a fixed point of (5) for $h = 0$. But this is not an oscillating solution. Thus, since locally the bifurcating branch is unique, the solutions belonging to this branch are constant, and (5) has no bifurcation point of oscillating solution at $h = 0$.

Now, consider the problem of finding 2-periodic solutions, that is, fixed points of F^2,

$$X = F^2(h, X). \tag{7}$$

Note that the eigenvalues of the linearization of F^k at zero, where k is an integer, are the eigenvalues of $D_X F$ to the power k. So finding bifurcation points of k-periodic solutions amounts to finding values of h for which the solutions of $p_h(r) = 0$ are the kth root of the unit. For (7), we want to know for which values of h, 1, or -1 are roots of $p_h(r)$.

The case $r = 1$ was studied above; consequently this case is without interest.

The case $r = -1$ gives $p_h(-1) = (1-ah)(-1)^p - (-1)^{p-1} - bh$. Two cases then occur:

(i) *p is odd.* Then $p_h(-1) = -(b-a)h - 2 < 0$, and never is equal to zero. So -1 is not an eigenvalue; therefore (7) has no periodic solution of period 2, lying in the vicinity of zero.

(ii) *p is even.* Then $p_h(-1) = 2 - (b+a)h$ and $p_h(-1) = 0$ if and only if $h = 2/(b+a)$. Thus -1 is a single root, and $(2/(b+a), 0)$ is a bifurcation point of nontrivial oscillating solutions.

Let $\mathscr{S}$ denote the closure of the set of nontrivial oscillating solutions of (7), $\mathscr{C}$ the maximal connected component in $\mathscr{S}$ containing $(2/(b+a), 0)$.

PROPOSITION III.1. *The bifurcating branch $\mathscr{C}$ is unbounded with respect to h.*

Proof. By the global bifurcation theorem [9], one can deduce that if $\mathscr{C}$ is bounded, then it meets another bifurcation point. But the only other bifurcation point is $(0, 0)$; and locally, at $(0, 0)$, the periodic solutions are constant. Thus they are not oscillating, and there exists a neighborhood V of $(0, 0)$ such that $\mathscr{C} \cap V = \{(0, 0)\}$. Indeed a connectedness argument leads to a contradiction. So $\mathscr{C}$ is unbounded.

B. *Hopf Bifurcation*

Now, we want to find h such that the roots r of $p_h(r) = 0$ are 1.

For that, let $r = e^{i\theta}$, and $\alpha = \cos(\theta)$.

Note that $\theta \neq k\pi$, where k is an integer, since this case was studied above, and $p_h(r)$ has the form

$$p_h(r) = (1 - ah)(r^2 - 2\alpha r + 1)(r^{p-2} + \beta_1 r^{p-3} + \cdots + \beta_{p-2}),$$

where β_j are given by

$$\begin{aligned}
&(1) \quad \beta_1 - 2\alpha = -1/(1 - ah),\\
&(2) \quad \beta_2 - 2\alpha\beta_1 + 1 = 0,\\
&(3) \quad \beta_j - 2\alpha\beta_{j-1} + \beta_{j-2} = 0; \qquad 3 \leqslant j \leqslant p-2,\\
&(4) \quad \beta_{p-3} = 2\alpha\beta_{p-2},\\
&(5) \quad \beta_{p-2} = -bh/(1 - ah).
\end{aligned} \tag{8}$$

The recurrence equation (8)(3) can be solved, and the solution is

$$\beta_j = Ce^{ij\theta} + De^{-ij\theta}, \qquad 3 \leqslant j \leqslant p-2, \tag{9}$$

where $i^2 = -1$ and C and D are real.

Let $j = p - 3$, then from (8) and (9) we obtain

$$(C - D)\sin((p-3)\theta) = 0 \tag{$*$}$$

Also, if $j = p - 2$ then

$$(C - D)\sin((p-2)\theta) = 0. \tag{$**$}$$

Suppose that $C \neq D$ in ($*$) and ($**$), then $\theta = m\pi$, where m is an integer. This contradicts the assumption on θ. So $C = D$, and it follows that

$$\cos((p-3)\theta) = \cos((p-3)\theta) + \cos((p-1)\theta).$$

And therefore

$$(p-1)\theta = \pi/2 + k\pi, \tag{10}$$

where k is an integer.

From (10) and $p_h(r) = 0$, we deduce that either

$$h = 0, \qquad \text{or} \qquad h = 2a/(b^2 - a^2).$$

The case $h = 0$ was studied above. Now, let $h = 2a/(b^2 - a^2)$. Then

$$p_h(r) = \left(1 - \frac{2a^2}{b^2 - a^2}\right) r^p - r^{p-1} - \frac{2ab}{b^2 - a^2} = 0,$$

has the solution

$$r = \frac{b^2 - a^2 \pm i2ab}{b^2 - 3a^2},$$

that is

$$\cos(\theta) = \frac{b^2 - a^2}{b^2 - 3a^2} \qquad \text{and} \qquad \sin(\theta) = \frac{2ab}{b^2 - 3a^2}.$$

But,

$$\cos^2(\theta) + \sin^2(\theta) = \frac{(b^2 - a^2)^2 + (2ab)^2}{(b^2 - 3a^2)^2} = 1,$$

and so either $a = 0$ or $a = \pm b$.

So, since $b > |a|$, $a \neq \pm b$ and thus $a = 0$, $\cos(\theta) = 1$, and $\sin(\theta) = 0$. Therefore $\theta = 2n\pi$, this contradicts the assumption on θ. So (10) is not valid, and we have the following result:

THEOREM III.2. *For p even, we have a bifurcation of Rabinowitz type. For p odd, and $p \geqslant 5$, there is no bifurcation of any kind.*

Looking at the results obtained independently by Hofbauer and Iooss, Kappel, and Schmitt, and our results, it seems that approximations by ordinary differential equations have better chances of imitating the exact equation. On the other hand, discrete approximations are a lot easier to handle, and the result by Haubauer and Iooss indicates that they may produce expected periodic solutions. There are two possible directions to explore:

(i) it may be that changing the approximation scheme will change the results and yield periodic solutions, or

(ii) it may be that when looking at approximations, the concept of periodic solutions is too rigid and it would be better to look at invariant manifolds.

References

1. O. Arino and R. Benkhalti, Periodic solutions for $\dot{x}(t) = \lambda f(x(t), x(t-1))$, *Proc. Roy. Soc. Edinburgh Sect. A* **109** (1988), 245–260.
2. O. Arino and P. Seguier, "Contribution à l'étude des comportements des solutions d'équation différentielle à retard par des méthodes de monotonie et de bifurcation." Thèse d'état, 1980.
3. R. Benkhalti, "Méthodes théoriques et numériques dans la détermination de phénomènes de bifurcation globale," Thèse de l'Université, 1986.
4. K. P. Hadler, Effective computation of periodic orbits and bifurcation diagrams in delay equations, *Numer. Math.* **34** (1980), 457–467.
5. D. Hankerson and A. Peterson, A classification of the solutions of a difference equation according to their behavior at infinity, *J. Math. Anal. Appl.* **136**, No. 1 (1988), 249–266.
6. M. W. Hirsch, Systems of differential equations which are competitive or cooperative, *SIAM J. Math. Anal.* **16** (1985), 423–439.
7. J. Hofbauer and G. Iooss, A Hopf bifurcation theorem for difference equations approximating a differential equation, *Mh Math.* **98** (1984), 99–113.
8. F. Kappel and K. Schmitt, Periodic solutions of systems of ordinary differential equations which approximate delay equations, *Differential Integral Equations* **1**, No. 2 (1988), 183–212.
9. P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7** (1971), 487–513.
10. E. M. Wright, A non-linear differential–difference equation, *J. Reine Angew. Math.* **194** (1955), 66–87.